PMM U.S.S.R., Vol. 46, No. 5, pp. 616-620, 1983.
Printed in Great Britain.

0021-8928/83 $7.50+0.00

UDC 531.36:62

INVESTIGATION OF A PARTICULAR MAPPING OF A CIRCLE*

V.N. BELYKH and L.V. LEBEDEVA

The one-dimensional mapping of the circle, the mathematical model of a discrete system of phase synchronization, is considered, and certain properties of the Poincaré rotation number and of the sequence of bifurcations which lead to complications in steady motions and to the emergence of stochasticity are established for it.

One of the possible causes of complications of motions in dynamic systems is associated with the destruction of invariant tori /1/. The mapping of sequence with respect to trajectories in the neighborhood of a two-dimensional invariant torus can be approximated by circle mapping. Destruction of the torus then corresponds the appearance of several numbers of rotation of that mapping.

1. Introduction and basic results. Evolution of nonlinear resonances and the sequence of bifurcations that lead to complications in the steady motions and emergence of stochasticity are considered here on the example of specific mapping of the form

$$\bar{\varphi} = \varphi + \alpha(\gamma - g(\varphi)) \equiv f(\varphi) \tag{1.1}$$

where α, γ are parameters, φ and $\bar{\varphi}$ are values of the angle variable at the n-th and $(n+1)$-st discrete instants of time, and $g(\varphi)$ is a periodic function.

Mapping (1.1) represents the mathematical model of a discrete system of phase synchronization /2,3/, hence our interest in it from the point of view of applied investigations. Note that mapping (1.1) for $g = \sin\varphi$ and small α was used for illustrating the theory of systems on a torus /4/, and for large α as the model of stochasticity origination /5/.

Let function $g(\varphi)$ satisfy the following Φ conditions: $g(\varphi)$ is continuous, periodic of period 2π and odd, the derivative $Dg(\varphi)$ admits a finite number of finite discontinuities at whose points it is additionally defined on the left and right by some values contained within its limits, $Dg(\varphi) > 0$ when $|\varphi| \leqslant \varphi_0$, $Dg(\varphi) < 0$ when $\varphi_0 < \varphi < 2\pi - \varphi_0$, $Dg(\varphi_0) = 0$, and $Dg(\varphi)$ does not increase for $\varphi \in [0, \varphi_0)$, $g(\varphi_0) = 1$.

The mapping trajectory f is to be understood as the sequence of points $\{f^n(\varphi)\}$, where f^n denotes the n-multiple mapping, and $n \in Z = \{0, \pm 1, \pm 2, \ldots\}$ is the discrete time. We shall call the mapping trajectory f a q/p cycle, if $\varphi_{n+p} = \varphi_n + 2\pi q$ (for $\varphi \in S^1$ this is a p-order cycle $\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+p} \pmod{2\pi} = \varphi_n$). Other utilized definition not adduced here appear in /4 – 16/.

We denote by Ω_0 the set of nonwandering trajectories of mapping the stright line $f: R^1 \to R^1$ $(\varphi \in R^1)$, and by Ω the mapping of circle $f: S^1 \to S^1$ $(\varphi \in S^1)$. Trajectories from the set Ω_0 will be called 0-trajectories, and those from the set $\Omega_\varphi = \Omega \setminus \Omega_0$ are φ-trajectories. In the motion of the mapping point on 0-trajectories (φ-trajectories) the over-all advance of phase φ is evidently zero (respectively, nonzero). We shall consider two cases of mapping f, viz. case H in which mapping f is reversible (one-to-one), i.e. when $1 - \alpha Dg(\varphi) > 0$ $\forall \varphi \in S^1$, and case E when f is irreversible, i.e. in the presence of the interval $I = \{\varphi \mid |\varphi| < \nu\}$, in which $1 - \alpha Dg(\varphi) < 0$, where $\pm\nu$ are zeros of the derivative Df.

In the H case, when f is a homeomorphism, the type of trajectories on the circle is determined by the Poincaré rotation number /4,6 – 8/.

$$r = \lim_{n\to\infty} (2\pi n)^{-1} f^n(\varphi) \tag{1.2}$$

which possesses the properties: r is a function of parameters, independent of φ, continuous which assumes rational (irrational) value $r = q/p$ then and only then when the mapping f has a q/p cycle (respectively, everywhere a compact quasi-periodic trajectory). In that case investigation of mapping f consists of the determination of q/p zones that are domains of

*Prikl.Matem.Mekhan.,Vol.46,No.5,pp.771-776,1982

parameters corresponding to stable rational numbers of rotation /7/ (irrational functions r are unstable), and to bifurcations of q/p cycles when their number varies within each zone.

When E with $\Omega_\varphi = 0$, mapping f is an endomorphism, i.e. it belongs to the class of irreversible mappings of the segment which attracts considerable attention (see /9-16/, and others) in connection with the problem of inducing stochasticity in dynamic systems. Phase rotation induced by mapping f leads in this case to the possibility of the simultaneous existence of several different q/p cycles, with consequent overlap of the q/p zones. The theory of the Poincaré rotation number does not work here, and the problem consists of studying the set of nonwandering trajectories $\Omega(f)$ and of its bifurcation at the change of parameters α, γ. Properties of that set can be complex /9/.

In the present investigation we establish the elementary properties of mapping f in the H case. The change of q/p zones at transition of mapping f from case H to case E with increasing α is investigated. Bifurcations that are interesting from the application point of view (see Fig.1) are illustrated on specific examples of $g = \sin\varphi$ and

$$g = \begin{cases} \varphi/\nu, & |\varphi| \leqslant \nu, \quad \varphi + \pi \pmod{2\pi} \\ (\pi - \varphi)/(\pi - \nu), & \nu < \varphi < 2\pi - \nu \end{cases} \tag{1.3}$$

In the E case particular attention is given to the parameter domain in which the set $\Omega(f) = \Omega_0$ (separated in Fig.1 by heavy lines) this domain is defined by mapping f on the attracting segment $O^+ \subset S^1$. We establish for points of that domain the existence of an infinite set of bifurcations of the $0/p$ cycles as α is increased, which follows in conformity with Sharkovskii's order /10/ up to the value $\alpha = \alpha_3^*$ (curve 3 in Fig.1) that defines the $0/3$ cycle bifurcation.

2. Preliminary analysis. When f is the mapping of a circle, i.e. when φ and $\bar{\varphi}$ of (1.1) are considered in $\operatorname{mod} 2\pi$, the substitution $\delta \to \delta + 2\pi k$, where $\delta = \alpha\gamma$, reduces (1.1) to its own form. (Mappings of (1.1) with $\delta = \delta^* + 2\pi k$ cover the f of "elevation" /8/ with various integral values of k). Then, taking into account the invariance of f to the substitution $\delta \to -\delta, \varphi \to -\varphi$, it is possible, without loss of generality, to consider $\alpha > 0$, $\delta \in [0, \pi]$, and, if mapping (1.1) with $\delta = \delta^*$ has a q/p cycle, it has a $(q + k)/p$-cycle when $\delta = \delta^* + 2\pi k$.

For $\gamma < 1$ the equation $\varphi = f(\varphi)$ has two solutions: $\varphi = \varphi_u \in (\varphi_0, \pi]$ which corresponds to the fixed point O_u (to $0/1$ cycle), unstable by virtue of $Df(\varphi_u) > 1$, and $\varphi = \varphi_s \in [0, \varphi_0]$ which corresponds to the fixed point O_s stable for $Df(\varphi_s) < 1$ and changes stability at the boundary $\Gamma_{12}^0 = \{(\alpha, \delta) \mid \alpha Dg(\varphi_s) = 2\}$. Let in the E case ν be the greater of the two roots of equation $1 - \alpha Dg(\varphi) = 0$ in the interval $(-\pi, \pi)$. When $\gamma < 1$, $\varphi_s > \nu$, $f(-\nu) < \varphi_u$ $\Omega_0 = \{\varphi_s\} \cup \{\varphi_u\}$, $\Omega_\varphi = 0$. From the Φ conditions we have that when

$$\gamma < 1, \quad f(-\nu) < \varphi_u, \quad \varphi_s < \nu \tag{2.1}$$

the trajectories of f reach segment $O^+ = \{\varphi \mid f(\nu) < \varphi < f(-\nu)\}$ and remain on it as $n \to \infty$. Thus under condition (2.1) mapping f has no φ-trajectories $\Omega_\varphi = 0$, $\Omega_0 = \Omega_0^+ \cup \{\varphi_u\}$, where $\Omega_0^+ \subset O^+$. When $f(-\nu) = \varphi_u$, a homoclinal trajectory is generated /15/ as $n \to \infty$ which approaches point $\varphi = \varphi_u$. A nontrivial hyperbolic set $\Omega_h \subset \Omega_\varphi$ exists under condition $f(-\nu) > \varphi_u$ /13/, with $\Omega_0 \neq 0$. When $\gamma = 1$, points O_u and O_s merge forming the semistable $(Df(\varphi_0) = 1)$ fixed point. When $\gamma > 1$ there are no fixed points $\Omega_0 = 0$. The phase rotates along any trajectory, since $f(\varphi) > \varphi$.

3. The H case. Let us consider the properties of the Poincare rotation number (1.2) of mapping f.

1) The rotation number r is a nondecreasing function of parameter γ. Indeed, by virtue of monotonicity of function $f(\varphi)$ the inequality $f^n(\varphi, \gamma) < f^n(\varphi, \gamma + h)$ is valid for $h > 0$, $n \geqslant 1$. Hence for any fixed φ_0

$$r(\gamma + h) = \lim_{n\to\infty} \frac{f^n(\varphi_0, \gamma + h)}{2\pi n} = r(\gamma) + \Delta, \quad \Delta > 0$$

2) When $\alpha > \delta$ or $\gamma < 1$, $r = 0$, since then the system has an 0/1-cycle.

3) When $\alpha = 0$ $\quad r = \lim\limits_{n\to\infty} \dfrac{\varphi + n\gamma}{2\pi n} = \dfrac{\gamma}{2\pi}$

4) When $\delta = \pi$ the rotation number $r = {}^1/_2$. Indeed, when $\delta = \pi$ points $\{k\pi\}$, $k \in Z$ form a 1/2-cylce, since then function $f(\varphi, \delta)$ assumes the value $f(k\pi, \pi) = (k + 1)\pi$. The rotation number $r = {}^1/_2$ is stable when $\delta = \pi$ if $(1 - \alpha Dg(0))(1 - \alpha Dg(\pi)) \neq 1$.

We introduce function $R(\varphi, \alpha, \delta) = f^p(\varphi, \alpha, \delta) - 2\pi q - \varphi$ whose zeros correspond to q/p cycles. Then domain of parameters D_{qp} in whose every point at which function R changes its sign for $\varphi \in S^1$ is a q/p zone. At the boundary of domain D_{qp} function R does not change its sign and vanishes at the points of q/p cycles bifurcation. The problem of contructing q/p zones thus reduces to finding zeros of function R and of their bifurcations.

Examples. 1°. When $g = \sin\varphi$ and $\alpha < 1$, (1.1) is a one-to-one mapping. In that case the q/p zones numerically determined for $q/p = 0, 1/6, 1/5, \ldots, 1$ are shown in Fig.1, and illustrate the above properties of the rotation number for $\alpha < 1$ (see also /4/).

2°. In the case of the piecewise-linear function $g(\varphi)$ of form (1.3) the boundaries of q/p zones can be obtained in explicit form.

4. The E case.

For parameters for which mapping f is irreversible two basic problems are of interest, viz. 1° determination of the structure of set $\Omega_0^+ \subset O^+$ in the parameter domain (2.1) and of its bifurcation, as parameter α is increased, and 2° determination of variation of the q/p zones as parameter α is increased, i.e. when passing from the H to the E case. Let us consider these problems.

1°. Generation of a 2-cycle mapping f takes place in the domain of parameters $\{\varphi_s \leqslant \nu, f^2(\nu) \leqslant \nu\}$ at stability change of the fixed point φ_s (at the boundary Γ_{12}°); it is also possible at the formation of an even-multiple root of equation $f^2(\varphi) = \varphi$. Then all points, except O_s and O_u and of unstable 2-cycles, are attracted by the stable 2-cycles. The form of mapping f on the invariant with respect to its segment $I_0 \subseteq O^+$ is shown in Fig.2,a for $f^2(\nu) = \nu$, $I_0 = [f(\nu), \nu]$, in Fig.2,b for $f(\nu) > -\nu$, $I_0 = [f(\nu), f^2(\nu)]$, in Fig.2,c for $f(\nu) = -\nu$, $I_0 = [-\nu, f(-\nu)]$, while in Fig.2,d for $f(\nu) < -\nu$, $I_0 = [f(\nu), f(-\nu)]$. The increase of α results in transitions from one type of mapping $f|_{I_0}$ to another: a-b-c-d.

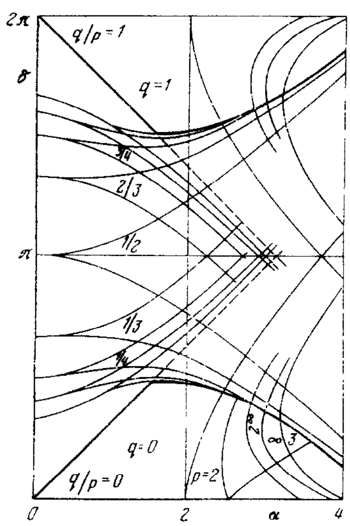

Fig.1

Mapping $f_0 = f|_{I_0}$ is isomorphic to some one-parameter set of mappings of segment $I = [0, 1]$ onto itself. Let F be an arbitrary set of continuous mappings of segment I, dependent on parameter α, and such that function $F(x, \alpha)$ satisfies condition Sh: $\alpha \in L = [\alpha_1, \alpha_3]$; $F(x, \alpha_1)$ has a fixed point that attracts all points of segment I, and $F(x, \alpha_3)$ has a 3-cycle. We denote by α_k^* the bifurcational value of parameter α such that $F(x, \alpha_k^*)$ has a k-cycle, while when $\alpha < \alpha_k^*$ there are no k-cycles. By virtue of continuity of function $F^k(x, \alpha) - x$ we have the following corollary of Sharkovskii's theorem /10/.

Corollary. Mapping $F(x, \alpha)$ has under condition Sh the sequence of bifurcation values α_k^* such that $\alpha_{k_2}^* \leqslant \alpha_{k_1}^*$, where k_2 is a number which follows any number k_1 in Sharkovskii's order

$$3 \succ 5 \succ 7 \succ \ldots \succ 3\cdot 2 \succ 5\cdot 2 \succ 7\cdot 2 \succ \ldots \succ 2^2 \succ 2 \succ 1 \qquad (4.1)$$

Indeed, if L_0 is the initial segment of series in which l_0 is the last element, then in the absence of l-cycles, where $l \in L_0$, the first to appear, as α is increased, is the l_0-cycle.

The following variants of bifurcation sequences are possible.

1) The bifurcation values α_k^* satisfy the strict inequality $\alpha_2^* < \alpha_4^* < \ldots < \alpha_5^* < \alpha_3^*$, with: a) k_1-cycle existing for any $\alpha > \alpha_{k_1}^*$, although it can bifurcate, and b) when $\alpha > \alpha_{k_1}^*$ k_1-cycles vanish and, then, reappear again.

2) There exists a segment $\{k_i, \ldots, k_j\}$, $i \neq j$ of series (4.1) such that bifurcation of cycles of respective periods $k = k_i, \ldots, k_j$ occur simultaneously when $\alpha = \alpha_{k_i}^* = \ldots = \alpha_{k_j}^*$.

All other types of bifurcation sequences of the cycle first creation are combinations of those indicated above.

The sufficient condition of separability of α_{2m}^*, $m = 1, 2, \ldots$ is, for instance, the invariance of F to Feigenbaum's transformation /11,14/. When $\alpha = \alpha_\infty^* = \lim_{n\to\infty} \alpha_{2^n}^*$, there exists a zero-measure Cantor set /14/ which attracts all points of segment I_0, except the points of unstable 2^n-cycles $(n = 1, 2, \ldots)$, and when $\alpha > \alpha_\infty^*$ the set Ω_0 is infinite and can be analyzed from ergodic theory point of view /9,11,12/. If, however, there exists a $2k_1$ different from power of two, such that $\alpha^*_{2^{n_1}} = \alpha^*_{2k_1}$, then α_∞^* is absent, and Ω_0 is infinite in the presence of the $2k_1$-cycles. An example of the latter is the mapping of form

$$F = \begin{cases} \alpha(x-0.5)+1, & 0 \leqslant x \leqslant 0.5 \\ 2(1-x), & 0.5 < x \leqslant 1 \end{cases} \qquad (4.2)$$

in which $\alpha_4^* = \alpha_{12}^* = 0.5$.

The considered here mapping (1.1) has a 3-cycle in the parameters domain $f^3(\nu) \geqslant \nu$, and by the corollary to Sharkovskii's theorem in the parameters domain $\{\gamma < 1,\ \varphi_s < \nu,\ f^3(\nu) < \nu\}$ it has bifurcation curves $\alpha = \alpha_k^*(\delta)$ ordered in conformity with (4.1). The strict inequalities $\alpha^\infty < \ldots < \alpha^*_{2m+1} < \ldots < \alpha_3^*$, where α^∞ is determined by one of the equations $f^3(\nu) = \varphi_s,\ f^2(-\nu) = \varphi_s$ and corresponds to the appearance of the trivial homoclinal trajectory (*), holds in the case of odd k. For even k, as in the case of function (4.2), $\alpha^*_{2n_1} = \alpha^*_{k_1}$ is possible for certain $n_1,\ k_1 = 2m$. Thus the sufficient conditions for Ω_0^+ to be infinite is the existence of cycles of odd multiplicity $f^3(\nu) > \varphi_s,\ f^2(-\nu) > \varphi_s$.

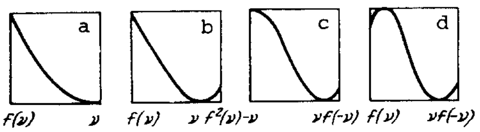

Fig.2

When $g = \sin\varphi$, the parameter domain in which $\Omega_\varphi = 0$ is defined by the inequalities

$$\gamma < 1,\quad \arcsin\gamma + \gamma\alpha < \pi + \arccos(1/\alpha) - \sqrt{\alpha^2-1} \qquad (4.3)$$

O^+ is determined by $\nu = \arccos(1/\alpha)$. As α increases inside domain (4.3) the fixed point φ_s looses its stability when $\delta = \sqrt{\alpha^2-4}$ (curve $(p=2)$ in Fig.1). Since $f'(\varphi_s) = -1$, a 0/2-cycle is generated from it. This is followed by doubling bifurcations (up to curve 2^∞ in Fig.1). When $\alpha > \alpha^*_{2^\infty}$ we have bifurcations that conform to Sharkovskii's order. Curve 3 $\alpha = \alpha_3^*$ in Fig.1 corresponds to bifurcation of the 0/3-cycle, while curve ∞ between curves 2^∞ and 3 corresponds to the generation of the homoclinal trajectory of point φ_s, i.e. to the appearance of the first odd cycle. The two curves that merge at $\delta = 0$ divide domain (4.3) into subregions on which mappings, shown qualitatively in Fig.2, obtain on segment O^+.

In the case of mapping (1.1), (1.3) the coordinates of fixed points are $\varphi_s = \gamma\nu$, $\varphi_u = \pi - \gamma(\pi-\nu)$, and condition $\Omega_\varphi = 0$ assumes the form $\gamma < 1,\ \gamma < (\pi - \alpha + \nu)(\pi + \alpha - \nu)^{-1}$. If $\alpha < 2\nu$, all points of the circle, except point φ_u, are attracted by the fixed point φ_s, and when $\alpha > 2\nu$, mapping f is elongating $|f'| > 1$ on segment $O^+ = \{\varphi \mid \delta - \alpha + \nu < \varphi \leqslant \delta + \alpha - \nu\}$, and, according to /12/ the limit set is a strainge attractor.

2°. It follows from /16/ that points of overlap of two q/p-zones also belong to the region of infinite number of q/p-zones. Construction of q/p-zones in the parameter domain where f is irreversible was carried out numerically on the example of mapping (1.1), $g = \sin\varphi$. The overlap of q/p-zones is shown in Fig.1 for several values of q/p. It is seen that the sequence of lower bounds of q/p-zones for $q/p < 1/2$ converges with increasing p to the curve $f(-\nu) = \varphi_u$, that corresponds to the generation of the nontrivial hyperbolic set Ω_h.

REFERENCES

1. AFRAIMOVICH V.S. and SHIL'NIKOV L.P., The ring principle in problems of interaction between two self-oscillating systems. PMM, Vol.41, No.4, 1977.
2. BELYKH V.N., On models of phase synchronization systems and their investigation. Dinamika Sistem, No.11, Gor'kii, Izd. Gor'k. Univ., 1976.
3. SHAKHTARIN B.I. and ARKHANGEL'SKII V.A., Dynamic characteristics of discrete systems of automatic phasing. Radiotekhnika i Elektronika. Vol.22, No.5, 1977.
4. ARNOL'D V.I., Supplementary Chapters of the Theory of Ordinary Differential Equations. Moscow, NAUKA, 1978.
5. ZASLAVSKII G.M. and RACHKO Kh.R., Characteristics of transition to turbulent motion. ZhETF, Vol.76, No.6, 1979.
6. MAIER A.G., The rough transformation of a circle into circle. Sc. Notes Gor'k. Univ., No. 12, 1939.
7. PLISS V.A., Nonlocal Problems of the Theory of Oscillations. Moscow—Leningrad, NAUKA, 1964.
8. NITETSKI E., Introduction to Differential Dynamics. Moscow, MIR, 1975.
9. IACOBSON M.V., On smooth mappings of circle onto itself. Matem. Sb., Vol.85, (127), 1971.
10. SHARKOVSKII A.N., Existence of cycles of continuous mapping of a straight line onto itself. Ukr. Matem. Zh., Vol.10, No.1, 1964.
11. SINAI Ia.G., On Feigenbaum's universality law. In: Nonlinear Waves. Gor'kii, Izd. Inst. Prikl. Fiz., Akad. Nauk SSSR, 1980.

*) F.C. Hoppensteadt and Y.M. Hymah, Courant Institute, N.Y., University preprint, 1975.

12. KOSIAKIN A.A., and SANDLER E.A., The ergodic properties of a class of piecewise-smooth transformations of a segment. Izv. VUZ, Matematika, No.3, 1972.
13. SENNIKOVSKII Ia. N., On the problem of smooth mappings of a circle with a denumerable set of periodic points. In: Methods of the Qualitative Theory of Differential Equations. Gor'kii, Izd. Gor'k. Univ., 1978.
14. BARKOVSKII Iu.S. and LEVIN G.M., On Cantor limit set. Uspekhi Matem. Nauk, Vol.39, No.2, 1980.
15. SHARKOVSKII A.N., On the problem of isomorphism of dynamic systems. In: Tr.V Internat. Conf. on Nonlinear Oscillations, Vol.2, Kiev, Izd. Inst. Matem. Akad. Nauk UkrSSR, 1970.
16. EFREMOVA L.S. and RAKHMANKULOV R.G., Theorems on the existence of periodic orbits of endomorphisms of circles. In: Differential and Integral Equations, Iss.4. Gor'kii, Izd. Gor'k. Univ., 1980.

Translated by J.J.D.